① $$\lim_{x\to0}\left(\frac{1+x^3}{1+x^2}\right)^{\frac{1}{\sqrt{x^2+1}-\cos x}} = e^{A} = e^{-\frac{2}{3}}$$

↖ VLSF(S) spoj. exp. v $-\frac{2}{3}$

$$A=\lim_{x\to0}\underbrace{\frac{\log()}{()-1}}_{\downarrow(*)\ 1}\cdot\frac{()-1}{\sqrt{x^2+1}-\cos x}\overset{AL}{=}\lim_{x\to0}\frac{1+x^3-(1+x^2)}{\underbrace{1+x^2}_{\downarrow 1}}\cdot\frac{\overbrace{\sqrt{x^2+1}}^{\to1}+\overbrace{\cos x}^{\to1}}{\frac{x^2+1-\cos^4 x}{x^2}\cdot x^2}\overset{AL}{=}2\lim_{x\to0}\frac{\cancel{x^2}(x-1)}{\cancel{x^2}\left(1+\frac{1-\cos x}{x^2}(1+\dots\cos^3 x)\right)}$$

$1-\cos^4x=(1-\cos x)(1+\cos x+\cos^2x+\cos^3x)$

$$\overset{AL}{=}2\cdot\frac{-1}{1+\frac12\cdot4}=-\frac{2}{3};$$ (*): VLSF(P), $F(y)=\frac{\log y}{y-1}\xrightarrow{y\to1}1$; $y=()=\frac{1+x^3}{1+x^2}\xrightarrow{x\to0}1$

$1\iff x^3=x^2\iff x\in\{0,1\}$

$\Rightarrow ()\neq1: x\in(-1,1)\setminus\{0\}$.

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② $$\lim_{x\to1}\frac{4^{2^x}-16}{\sqrt{1-\cos(2\pi x)}}=\lim_{x\to1}16\cdot\frac{4^{2^x-2}-1}{\sqrt{1-\cos(2\pi x)}}\overset{16}{=}\lim_{x\to1}\underbrace{\frac{e^{\log4(2^x-2)}-1}{\log4(2^x-2)}}_{\to1\,(*)}\cdot\log4\cdot\underbrace{\frac{e^{\log2(x-1)}-1}{\log2(x-1)}}_{\downarrow(**)\ 1}\cdot$$

$$\log2(x-1)\cdot\underbrace{\sqrt{\frac{(2\pi(x-1))^2}{1-\cos(2\pi(x-1))}}}_{\downarrow\ \sqrt2\,(***)}\cdot\frac{1}{2\pi|x-1|}=\lim_{x\to1}\frac{32\log2\cdot\log4\cdot\sqrt2}{2\pi}\cdot\frac{x-1}{|x-1|}$$

$$\Rightarrow\lim_{x\to1\pm}\frac{4^{2^x}-16}{\sqrt{1-\cos(2\pi x)}}=\pm16\sqrt2\log(2)\log(4)/\pi\Rightarrow$$ limita neex.

(*) VLSF(P): $F(y)=\frac{e^y-1}{y}\xrightarrow{y\to0}1$; $y=\log4(2^x-2)\xrightarrow{x\to1}0$ ↑prostá fce ⇒ (P)

(**) VLSF(P): — // — $y=\log2(x-1)\xrightarrow{x\to1}0$, $\neq0: x\in\mathbb{R}\setminus\{1\}$

(***) VLSF(S): záměna odmocniny a limity – spoj. √ v 2

VLSF(P): $F(y)=\frac{y^2}{1-\cos y}\xrightarrow{y\to0}2$; $y=2\pi(x-1)\xrightarrow{x\to1}0$, $\neq0: x\in\mathbb{R}\setminus\{1\}$.

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③ $$\lim_{x\to0}\frac{\sqrt{1+\cos x\log(\cos x)}-\sqrt{1+\log(\cos x)}}{\sqrt[3]{\operatorname{tg}x}-\sqrt[3]{\sin x}}\cdot\frac{\sqrt[3]{x}}{\sin^2x}=\lim_{x\to0}\underbrace{\frac{\log(\cos x)}{\cos x-1}}_{\downarrow(\text{dříve})\ 1}\underbrace{\frac{(\cos x-1)^2}{x^4}}_{\frac14}\cdot\underbrace{\frac{x^2}{\sin^2x}}_{1}\cdot\frac{x^2}{\underbrace{\sqrt{}}_{1}+\underbrace{\sqrt{}}_{1}}$$

spoj. √ v 1 VLSF(S)

$$\frac{(\sqrt[3]{\operatorname{tg}x})^2+\dots+(\sqrt[3]{\sin x})^2\cdot\sqrt[3]{x^2}\cdot\sqrt[3]{x}}{\sqrt[3]{x^2}}\overset{AL}{=}\frac38\lim_{x\to0}\frac{\cancel{x}^3}{\cancel{x}^3\cdot\underbrace{\frac{\sin x}{x}}_{\downarrow 1}\underbrace{\frac{(1-\cos x)}{x^2}}_{\downarrow \frac12}}\overset{AL}{=}\frac34$$

$\frac{\sin x}{\cos x}-\sin x$

konv. k 3 (VLSF(S) spoj. $\sqrt[3]{}$ v 1)

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④ $$\lim_{x\to0}\frac{1-\cos x\cos2x\cos3x}{1-\cos x}=\lim_{x\to0}\frac{1-\cos x+\cos x-\cos x\cos2x+\cos x\cos2x-\cos x\cos2x\cos3x}{1-\cos x}=$$

$$\lim_{x\to0}1+\cos x\frac{1-\cos2x}{1-\cos x}+\cos x\cos2x\frac{1-\cos3x}{1-\cos x}\overset{AL}{=}1+\lim_{x\to0}\frac{x^2}{1-\cos x}\cdot\underbrace{\frac{1-\cos2x}{(2x)^2}}_{\downarrow\ \frac12(*)}\cdot4+\lim_{x\to0}\frac{x^2}{1-\cos x}\cdot\underbrace{\frac{1-\cos3x}{(3x)^2}}_{\downarrow\ \frac12(*)}\cdot9$$

$$\overset{AL}{=}1+2\cdot\frac12\cdot4+2\cdot\frac12\cdot9=\underline{14}$$

(*) VLSF(P) $a\neq0$: $F(y)=\frac{1-\cos y}{y^2}\xrightarrow{y\to0}\frac12$; $y=ax\xrightarrow{x\to0}0$, $\neq0: x\in\mathbb{R}\setminus\{0\}$.

⑤ $$\lim_{x\to1}\frac{\sin^2(\pi 2^x)}{\log(\cos(\pi\cdot 4^x))} \overset{\uparrow}{=} \lim_{x\to1}\underbrace{\frac{\sin^2(\pi(2^x-2))}{(\pi(2^x-2))^2}}_{\Downarrow 1^{(*)}}\cdot 4\pi^2\cdot\underbrace{\left(\frac{e^{\log 2(x-1)}-1}{\log 2(x-1)}\right)^2}_{\Downarrow 1^{(**)}}\log^2 2\,(\cancel{x-1})^2\cdot$$

$\sin^2, \cos$ jsou 2π-per.

$$\underbrace{\frac{\cos(\,)-1}{\log(\cos(\pi(4^x-4)))}}_{\Downarrow 1^{(+)}}\cdot\underbrace{\frac{(\,)^2}{\cos(\,)-1}}_{\Downarrow -2^{(+++)}}\cdot\frac{1}{\pi^2\cdot 16}\cdot\left(\frac{\cancel{e^{\log 4(x-1)}-1}}{\cancel{\log 4(x-1)}}\right)^2\underbrace{\left(\frac{\log 4(x-1)}{e^{\log 4(x-1)}-1}\right)^2}_{\Downarrow 1^{(++)}}\cdot$$

$$\frac{1}{\log^2 4(\cancel{x-1})^2} = -\frac{1}{8}$$

$\uparrow \log 4 = 2\log 2$

(*) VLSF(P) $F(y)=\dfrac{\sin y}{y}\xrightarrow{y\to0}1;\ y=\pi(2^x-2)\xrightarrow{x\to1}0$
$\nwarrow$ rostoucí $\Rightarrow$ (P)

(**) VLSF(P) $F(y)=\dfrac{e^y-1}{y}\xrightarrow{y\to0}1;\ y=\log 2(x-1)\xrightarrow{x\to1}0$
$\neq 0: x\in\mathbb{R}\setminus\{0\}$

(+) VLSF(P) $F(y)=\dfrac{y-1}{\log y}\xrightarrow{y\to1}1;\ y=\cos(\,)\xrightarrow{x\to1}1\Leftarrow(\,)\xrightarrow{x\to1}0$
$(\,)=\pi(4^x-4)\neq0: x\in\mathbb{R}\setminus\{0\}$, $(\,)\to0\Rightarrow\exists\delta>0:$
$x\in(-\delta,\delta): (\,)\neq 2k\pi, k\neq0 \Rightarrow x\in(-\delta,\delta)\setminus\{0\}:$
$\cos(\,)\neq1 \Rightarrow$ (P).

(++) VLSF(P) $F(y)=\dfrac{y}{e^y-1}\xrightarrow{y\to0}1;\ y=\log 4(x-1)$ $\Leftarrow$ rostoucí $\Rightarrow$ (P)
$\Downarrow^{x\to1}$ 0

(+++) VLSF(P) $F(y)=\dfrac{y^2}{\cos y-1}\xrightarrow{y\to0}-2;\ y=(\,)=\pi(4^x-4)\xrightarrow{x\to1}0$
$\neq 0: x\in(-\delta,\delta)\setminus\{0\}$

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⑥ $$\lim_{x\to0}\frac{(\cos x)^x-\sqrt{1+\sin^3 x}}{x^3}\overset{AL}{=}\lim_{x\to0}\frac{(\cos x)^x-1}{x^3}+\lim_{x\to0}\frac{1-\sqrt{1+\sin^3 x}}{x^3}=\lim_{x\to0}\underbrace{\frac{e^{x\log(\cos x)}-1}{x\log(\cos x)}}_{\Downarrow 1^{(*)}}\cdot$$

$$x\cdot\underbrace{\frac{\log(\cos x)}{\cos x-1}}_{\Downarrow 1}\cdot\underbrace{\frac{\cos x-1}{x^2}}_{-\frac12}\cdot x^2\cdot\frac{1}{x^3}+\lim_{x\to0}\underbrace{\frac{-\sin^3 x}{x^3}}_{\Downarrow -1}\cdot\frac{1}{\underbrace{1+\sqrt{1+\sin^3 x}}_{\Downarrow 1}}$$

$\Downarrow$ VLSF(S) spoj. $\sqrt{\ }$ v 1.

$$=-\frac12+\left(-\frac12\right)=-1$$

(*) VLSF(P) $F(y)=\dfrac{e^y-1}{y}\xrightarrow{y\to0}1;\ y=x\log(\cos x)\xrightarrow{x\to0}0$
$\neq 0 \quad x\neq 2k\pi$
$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\setminus\{0\}$ $\cos x>0$

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⑦ $$\lim_{x\to1}\frac{x^{x-1}+\sqrt[3]{\cos\pi x}}{\log^2 x}\overset{AL}{=}\lim_{x\to1}\underbrace{\frac{(x-1)^2}{\log^2 x}}_{\Downarrow 1}\cdot\frac{1}{(\cancel{x-1})^2}\cdot\left(\underbrace{\frac{e^{(x-1)\log x}-1}{(x-1)\log x}}_{\Downarrow 1}\cdot(\cancel{x-1})\cdot\underbrace{\frac{\log x}{x-1}}_{\Downarrow 1}(\cancel{x-1})+\right.$$

$\cos(\lambda+\pi)=-\cos\lambda$

$$\underbrace{\frac{1-\cos(\pi(x-1))}{(\pi(x-1))^2}}_{\Downarrow\frac12}\cdot\frac{\pi^2(\cancel{x-1})^2}{\underbrace{1+\sqrt[3]{\ }}_{\Downarrow 1}+\underbrace{(\sqrt[3]{\ })^2}_{\Downarrow 1}}\overset{AL}{=}1+\frac{\pi^2}{6}$$

$\Downarrow 1 \Leftarrow$ VLSF(S) spoj. $\sqrt[3]{\ }$ v 1.

WOLFRAM CHYBA

(8) Pozn: $\lim_{t\to 0_+} (\log t)\, t^{\beta} = 0 : \beta > 0 \Leftarrow$ substituce $t = \frac{1}{y}$ a škála nebo l'Hospital.

$$\lim_{t\to 0_+} \frac{\log t}{\frac{1}{t^{\beta}}} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{\text{l'Hosp.}}{=}} \lim_{t\to 0_+} \frac{\frac{1}{t}}{-\beta t^{-\beta-1}} = \lim_{t\to 0_+} -\beta t^{\beta} = 0$$

$$\lim_{x\to 0_+} (e^{x}-1)^{\frac{\lg^2 x}{x^{\alpha}}} = e^{A}$$

$$A = \lim_{x\to 0_+} \log(e^{x}-1)\frac{\lg^2 x}{x^{\alpha}} = \begin{cases} -\infty : \alpha = 2 \\ -\infty : \alpha > 2 \\ \text{y}(*) : \alpha < 2 \end{cases}$$

$$(*) = \lim_{x\to 0_+} \underbrace{\log(e^{x}-1)\cdot(e^{x}-1)^{2-\alpha}}_{\downarrow (**)\ 0} \cdot \underbrace{\frac{x^{2-\alpha}}{(e^{x}-1)^{2-\alpha}}}_{\downarrow (+)\ 1} \cdot \underbrace{\frac{\lg^2 x}{x^2}}_{\downarrow\ 1} = 0$$

$(**)$ $F(y) = \log y \cdot y^{2-\alpha} \to 0 : y \to 0$ ← Pozn.

$y = e^{x} - 1 \underset{x\to 0}{\to} 0$

$\neq 0 : x \in \mathbb{R}\setminus\{0\}$

$(+)$ ~~F(y)~~ = VLSF (S) spoj. s^{α} v bodě 1, $\frac{\cancel{\lg}\, x}{e^{x}-1} \overset{x\to 1}{\to} 1$

$1^{\alpha} = 1.$